Prime Gamma Rings with Centralizing and Commuting Symmetric Bi derivations

C. Jaya Subba Reddy[1], C. Venkata Sai Raghavendra Reddy[2] K. Nagesh[3]

[1]Associate Professor, Department of Mathematics, S.V.University, Tirupati-517502, A.P., India.

[2]Amrita Vishwa Vidyapeetham, Bengaluru Campus, Bengaluru, Karnataka-560035, India.
[3]Research Scholar, Rayalaseema University, Kurnool, Andhra Pradesh, India.

Abstract: Let M be a prime Γ-ring satisfying a certain assumption (A) and D a nonzero derivation on M. If $D(.,.): MXM \to M$ is a symmetric bi derivation and d is trace of D on a left ideal J of M such that d is centralizing and commuting on J, then M is commutative.

Key words: Prime Γ-ring, Centralizing and Commuting, Derivation, Symmetric, Symmetric bi derivation , Bi additive mapping, Symmetric bi additive mapping, Trace.

Preliminaries: Let M and Γ be additive abelian groups. If there exists a mapping $(x, \alpha, y) \to x\alpha y$ of $M \times \Gamma \times M \to M$, which satisfies the conditions
(i) $x\alpha y \in M$
(ii) $(x + y)\alpha z = x\alpha z + y\alpha z, x(\alpha + \beta)z = x\alpha z + x\beta z, x\alpha(y + z) = x\alpha y + x\alpha z$
(iii) $(x\alpha y)\beta z = x\alpha(y\beta z)$ for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$, then M is called a Γ-ring.
Every ring M is a Γ-ring with $M = \Gamma$. However a Γ-ring need not be a ring. Let M be a Γ-ring. Then an additive subgroup U of M is called a left (right) ideal of M if $M\Gamma U \subset U(U\Gamma M \subset U)$. If U is both a left and a right ideal, then we say U is an ideal of M. Suppose again that M is a Γ-ring. Then M is said to be a 2-torsion free if $2x = 0$ implies $x = 0$ for all $x \in M$. An ideal P_1 of a Γ-ring M is said to be prime if for any ideals A and B of M, $A\Gamma B \subseteq P_1$ implies $A \subseteq P_1$ or $B \subseteq P_1$. An ideal P_2 of a Γ-ring M is said to be semiprime if for any ideal U of M, $U\Gamma U \subseteq P_2$ implies $U \subseteq P_2$. A Γ-ring M is said to be prime if $a\Gamma M\Gamma b = (0)$ with $a, b \in M$, implies $a = 0$ or $b = 0$ and semiprime if $a\Gamma M\Gamma a = (0)$ with $a \in M$ implies $a = 0$. Furthermore, M is said to be commutative Γ-ring if $x\alpha y = y\alpha x$ for all $x, y \in M$ and $\alpha \in \Gamma$. Moreover, the set $Z(M) = \{x \in M: x\alpha y = y\alpha x$ for all $y \in M$ and $\alpha \in \Gamma\}$ is called the centre of the Γ-ring M. If M is a Γ-ring, then $[x, y]_\alpha = x\alpha y - y\alpha x$ is known as the commutator of x and y with respect to α, where $x, y \in M$ and $\alpha \in \Gamma$. We make the basic commutator identities:
$[x\alpha y, z]_\beta = [x, z]_\beta \alpha y + x\alpha[y, z]_\beta$ and $[x, y\alpha z]_\beta = [x, y]_\beta \alpha z + y\alpha[x, z]_\beta$, for all $x, y \in M$ and $\alpha \in \Gamma$.

We consider the following assumption:
(A).................$x\alpha y\beta z = x\beta y\alpha z$, for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$. An additive mapping $D: M \to M$ is called a derivation if $D(x\alpha y) = D(x)\alpha y + x\alpha D(y)$ holds for all $x, y \in M$ and $\alpha \in \Gamma$. A mapping d is said to be commuting on a left ideal J of M if $[d(x), x]_\alpha = 0$ for all $x \in J$ and $\alpha \in \Gamma$ and d is said to be centralizing if $[d(x), x]_\alpha \in Z(M)$ for all $x \in J$ and $\alpha \in \Gamma$. A mapping $D(.,.): RXR \to R$ is called symmetric if $D(x, y) = D(y, x)$, for all $x, y \in R$.A mapping $d: R \to R$

defined by $d(x) = D(x, x)$ is called the trace of D, where $D(.,.): RXR \rightarrow R$ is a symmetric mapping. It is obvious that, if $D(.,.): RXR \rightarrow R$ is a symmetric mapping which is also bi-additive (i.e. additive in both arguments), then the trace d of D satisfies the relation $d(x + y) = d(x) + d(y) + 2D(x, y)$, for all $x, y \in R$. If $D(.,.)$ is bi-additive and satisfies the identities $D(x\beta y, z) = D(x, z)\beta y + x\beta D(y, z)$ and $D(x, y\beta z) = D(x, y)\beta z + y\beta D(x, z)$, for all $x, y, z \in R, \beta \in \Gamma$. Then $D(.,.)$ is called a symmetric bi-derivation.

Introduction: The concept of the Γ-ring was first introduced by Nobusawa [15] and also shown that Γ-rings, more general than rings. Bernes [1] weakened slightly the conditions in the definition of Γ-ring in the sense of Nobusawa. Bresar [2] studied centralizing mappings and derivations in prime rings. Kyuno [10] , Luh [11], [12], Hoque and Paul [5],[6] and others were obtained a large numbers of important basic properties of Γ-rings in various ways and determined some more remarkable results of Γ-rings. Ceven [3] studied on Jordan left derivations on completely prime Γ-rings. Mayne [14] have developed some remarkable result on prime rings with commuting and centralizing. Jaya Subba Reddy.C et.al [8] studied centralizing and commutating left generalized derivation on prime ring is commutative. The concept of a symmetric bi-derivation has been introduced by Maksa in [13]. J.Vukman [17],[18] investigated symmetric bi-derivations on prime and semiprime rings. Jaya Subba Reddy.C et.al[9] has studied some results on symmetric reverse bi-derivations on prime rings. M.S.Yenigul and N.Argac [16] studied few results on ideals and symmetric bi-derivations of prime and semiprime rings. Hoque and Paul[7] studied prime gamma rings with centralizing and commuting generalized derivations is a commutative. In this paper, we extended some results on prime gamma rings with centralizing and commuting symmetric bi derivation is a commutative.

Some preliminary results

We have to make some use of the following well-known results:

Remark 1: Let M be a prime Γ-ring. If $a\alpha b \in Z(M)$ with $0 \neq a \in Z(M)$, then $b \in Z(M)$.

Remark 2: Let M be a prime Γ-ring and J a nonzero left ideal of M. If D is a nonzero derivation on M, then D is also a nonzero on J.

Remark 3: Let M be a prime Γ-ring and J a nonzero left ideal of M. If J is commutative, then M is also commutative.

Lemma 1: Suppose M is a prime Γ-ring satisfying the assumption (A) and $D(.,.): MXM \rightarrow M$ be a symmetric bi derivation. For an element $a \in M$, if $a\alpha D(x, y) = 0$ for all $x, y \in M$ and $\alpha \in \Gamma$, then either $a = 0$ or $D = 0$.

Proof: By our assumption, $a\alpha D(x, y) = 0$, for all $x, y \in M, \alpha \in \Gamma$.

We replacing x by $x\beta z$ in above equation then, we get

$a\alpha D(x\beta z, y) = 0$

$a\alpha\big(D(x, y)\beta z + x\beta D(z, y)\big) = 0$

$a\alpha D(x,y)\beta z + a\alpha x\beta D(z,y) = 0$

$a\alpha x\beta D(z,y) = 0$, for all $x,y,z \in M$, $\alpha,\beta \in \Gamma$.

If D is nonzero, that is, if $D(z,y) \neq 0$ for some $z,y \in M$, then by definition of prime Γ-ring, $a = 0$.

Lemma 2: Suppose M is a prime Γ-ring satisfying the assumption (A) and J a nonzero left ideal of M. If M has a symmetric bi derivation $D(.,.)\colon MXM \to M$ which is zero on J, then D is zero on M.

Proof: By the hypothesis, $D(J,U) = 0$.

Replacing J by $M\Gamma$J, we get

$D(M\Gamma \mathrm{J}, U) = 0.$

$D(M,U)\Gamma \mathrm{J} + \mathrm{M}\Gamma \mathrm{D}(\mathrm{J},\mathrm{U}) = 0$

$D(M,U)\Gamma \mathrm{J} = 0.$

Hence by Lemma1, D must be zero, Since J is nonzero.

Lemma 3[7]: Suppose M is a prime Γ-ring satisfying the assumption (A) and J a nonzero left ideal of M. If J is commutative on M, then M is commutative.

Lemma 4: Suppose M is a prime Γ-ring and $D(.,.)\colon MXM \to M$ be an symmetric bi additive mapping with trace d of D. If d is centralizing on a left ideal $J$of M, then $f(a) \in Z(M)$ for all $a \in J \cup Z(M)$.

Proof: d is a centralizing a on left ideal J of M, we have

$[d(x),x]_\alpha \in Z(M)$ for all $x \in J$ and $\alpha \in \Gamma$.

By linearization, we get

$x,y \in J \Longrightarrow x+y \in J$, for all $\alpha \in \Gamma$.

$[d(x+y), x+y]_\alpha \in Z(M)$

$[d(x) + d(y) + 2D(x,y), x+y]_\alpha \in Z(M)$

$[d(x),x]_\alpha + [d(x),y]_\alpha + [d(y),x]_\alpha + [d(y),y]_\alpha + 2[D(x,y),x]_\alpha + 2[D(x,y),y]_\alpha \in Z(M)$

$[d(x),y]_\alpha + [d(y),x]_\alpha + 2[D(x,y),x]_\alpha + 2[D(x,y),y]_\alpha \in Z(M)$

If $x \in Z(M)$, then above equation, we get

$[d(y),x]_\alpha = 0, 2[D(x,y),x]_\alpha = 0$

$[d(x),y]_\alpha + 2[D(x,y),y]_\alpha \in Z(M)$

Replacing y by $y+z$ in above equation, we get

$[d(x),y+z]_\alpha + 2[D(x,y+z),y+z]_\alpha \in Z(M)$

$[d(x),y]_\alpha + [d(x),z]_\alpha + 2[D(x,y),y+z]_\alpha + 2[D(x,z),y+z]_\alpha \in Z(M)$

$[d(x),y]_\alpha + [d(x),z]_\alpha + 2[D(x,y),y]_\alpha + 2[D(x,y),z]_\alpha + 2[D(x,z),y]_\alpha + 2[D(x,z),z]_\alpha \in Z(M)$

$2[D(x,z),y]_\alpha + 2[D(x,y),z]_\alpha \in Z(M).$ (1)

If $Z \in Z(M)$, then equation (1) becomes

$2[D(x,z),y]_\alpha \in Z(M)$

Replacing z by x in above equation, we get

$2[D(x,x),y]_\alpha \in Z(M)$

$2[d(x),y]_\alpha \in Z(M)$

$[d(x),y]_\alpha \in Z(M).$

Replacing y by $d(x)\beta y$ in above equation, we get

$[d(x),d(x)\beta y]_\alpha \in Z(M).$

$d(x)\beta[d(x),y]_\alpha + [d(x),d(x)]_\alpha \beta y \in Z(M).$

$d(x)\beta[d(x),y]_\alpha \in Z(M)$. If $[d(x),y]_\alpha = 0.$

Then $d(x) \in C_{\Gamma M}(J).$

The centralizer of J in M and hence $d(x) \in Z(M)$. Otherwise, if $[d(x),y]_\alpha \neq 0$, remark 1 follows that $d(x) \in Z(M)$.

Theorem 1: Let M be a prime Γ-ring satisfying the assumption (A) and D a nonzero derivation on M. If $D(.,.): MXM \to M$ is a symmetric bi derivation and d is trace of D on a left ideal J of M such that d is commuting on J, then M is commutative.

Proof: Since d is commuting on J, we have

$[d(x),x]_\alpha = 0$, for all $x \in J$ and $\alpha \in \Gamma$.

Replacing x by $x+y$ in above equation, we get

$[d(x+y),x+y]_\alpha = 0$

$[d(x)+d(y)+2D(x,y),x+y]_\alpha = 0$

$[d(x),x]_\alpha + [d(x),y]_\alpha + [d(y),x]_\alpha + [d(y),y]_\alpha + 2\,[D(x,y),x]_\alpha + 2[D(x,y),y]_\alpha = 0$

$[d(x),y]_\alpha + [d(y),x]_\alpha + 2\,[D(x,y),x]_\alpha + 2[D(x,y),y]_\alpha = 0.$ (2)

Replacing x by $-x$ in equation (2), we get

$[d(-x),y]_\alpha + [d(y),-x]_\alpha + 2\,[D(-x,y),-x]_\alpha + 2[D(-x,y),y]_\alpha = 0$

$[d(x),y]_\alpha - [d(y),x]_\alpha + 2\,[D(x,y),x]_\alpha - 2[D(x,y),y]_\alpha = 0.$ (3)

Adding equations (2) and (3), we get

$[d(x),y]_\alpha + 2\,[D(x,y),x]_\alpha = 0.$ (4)

Replacing y by $x\beta y$ in equation (4), we get

$[d(x),x\beta y]_\alpha + 2\,[D(x,x\beta y),x]_\alpha = 0$

$[d(x),x]_\alpha \beta y + \mathrm{x}\beta\,[d(x),y]_\alpha + 2\,[D(x,x)\beta y + x\beta D(x,y),x]_\alpha = 0$

$[d(x),x]_\alpha \beta y + \mathrm{x}\beta\,[d(x),y]_\alpha + 2\,[d(x)\beta y,x]_\alpha + 2\,[\mathrm{x}\beta\, D(x,y),x]_\alpha = 0$

$[d(x),x]_\alpha \beta y + \mathrm{x}\beta\,[d(x),y]_\alpha + 2d(x)\beta[y,x]_\alpha + 2[d(x),x]_\alpha \beta y + 2\mathrm{x}\beta\,[\,D(x,y),x]_\alpha +$
$2[x,x]_\alpha \beta D(x,y) = 0$

$\mathrm{x}\beta\,[d(x),y]_\alpha + 2d(x)\beta[y,x]_\alpha + 2\mathrm{x}\beta\,[\,D(x,y),x]_\alpha = 0$

$\mathrm{x}\beta(\,[d(x),y]_\alpha + 2\,[\,D(x,y),x]_\alpha) + 2d(x)\beta[y,x]_\alpha = 0$

Using equation (4) in above equation, we get

$d(x)\beta[y,x]_\alpha = 0$

Replacing y by $x\gamma r$ in above equation, we get

$d(x)\beta[x\gamma r,x]_\alpha = 0$

$d(x)\beta x\gamma[r,x]_\alpha + d(x)\beta[x,x]_\alpha \gamma r = 0$

$d(x)\beta x\gamma[r,x]_\alpha = 0$, for some $x \in J, r \in M$, and $\alpha,\beta,\gamma \in \Gamma$.

Since M is a prime Γ-ring $d(x) = 0$ or $[r,x]_\alpha = 0$.

$\Rightarrow \quad D(x,x) = 0.$

Since D is nonzero derivation on M, then by lemma 2, D is nonzero on J.

Suppose $D(x,x) \neq 0$ for some $x \in J$, then $x \in Z(M)$.

Let $z \in J$ with $z \neq Z(M)$. Then $D(z,z) = 0$ and $x + z \notin Z(M)$, that is, $D(x+z) = 0$ and so $D(x,x) = 0$, which is a contradiction. Thus $z \in Z(M)$ for all $z \in J$. Hence J is commutative and lemma 3, we get M is commutative.

Theorem 2: Let M be a prime Γ-ring satisfying the assumption (A) and J a left ideal of M with $J \cap Z(M) \neq 0$. If $D(.,.): MXM \to M$ is a symmetric bi derivation on trace d of D, such that d is commuting on J, then M is commutative.

Proof: We claim that, $Z(M) \neq 0$ because of d is commuting on J and the proof is complete.

Now from equation (1), we get

$$2[D(x,z),y]_\alpha + 2[D(x,y),z]_\alpha \in Z(M)$$

$$\Longrightarrow [D(x,z),y]_\alpha + [D(x,y),z]_\alpha \in Z(M)$$

We replace y by $z\beta w$ with $0 \neq z \in Z(M)$, then we get

$$[D(x,z),z\beta w]_\alpha + [D(x,z\beta w),z]_\alpha \in Z(M)$$

$$[D(x,z),z]_\alpha \beta w + z\beta[D(x,z),w]_\alpha + [D(x,z)\beta w + z\beta D(x,w),z]_\alpha \in Z(M)$$

$$[D(x,z),z]_\alpha \beta w + z\beta[D(x,z),w]_\alpha + [D(x,z)\beta w,z]_\alpha + [z\beta D(x,w),z]_\alpha \in Z(M)$$

Since $z \in Z(M) \Longrightarrow d$ is a centralizer on J.

$$z\beta[D(x,z),w]_\alpha \in Z(M).$$

As z is nonzero, remark 1 follows that $[D(x,z),w]_\alpha \in Z(M)$. This implies that D is centralizing on J and hence we conclude that M is commutative.

References

[1] Bernes.W.E: On the Γ-rings of Nobusawa, Pacific J. Math., Vol. 18,(1966),411-422.

[2] Bresar.M: Centralizing Mappings and Derivations in Prime Rings, Journal of Algebra, Vol. 156,(1993), 385-394.

[3] Ceven.Y: Jordan left derivations on completely prime gamma rings, C. U. Fen-Edebiyat Fakultesi, Fen Bilimleri Dergisi(2002)Cilt 23 Sayi 2.

[4] Hoque.M.F and Paul.A.C: On Centralizers of Semiprime Gamma Rings, International Mathematical Forum, Vol. 6, 13(2011),627-638.

[5] Hoque.M.F and Paul.A.C: An Equation Related to Centralizers in Semiprime Gamma Rings, Annals of Pure and Applied Mathematics, Vol. 1,1(2012),84-90.

[6] Hoque.M.F and Paul.A.C: Centralizers on Semiprime Gamma Rings, Italian J. of Pure and Applied Mathematics, Vol. 30,(2013),289-302.

[7] Hoque.M.F and Paul.A.C: Prime gamma rings with centralizing and commuting generalized Derivations, International Journal of Algebra, Vol.7,13(2013),645-651.

[8] Jaya Subba Reddy.C, Mallikarjuna Rao.S and Vijaya Kumar.V: Centralizing and commuting left generalized derivations on prime rings, Bulletin of Mathematical Science and Applications,Vol.11,(2015),1-3.

[9] Jaya Subba Reddy.C et.al: Symmetric reverse bi-derivations on prime rings, Research J. Pharm. and Tech, Vol. 9, 9 (2016), 1496-1500.

[10] Kyuno.S: On prime Gamma ring, Pacific J.Math.,Vol. 75,(1978),185-190.

[11] Luh.J: On the theory of simple Gamma rings, Michigan Math.J, Vol. 16,(1969),65-75.

[12] Luh.J: A note on commuting automorphism of rings, Amer. Math. Monthly, Vol. 77,(1970), 61-62.

[13] Maksa.G: On the trace of symmetric bi-derivations, C.R. Math. Rep. Sci. Canada Vol. 9,(1987), 303-307.

[14] Mayne.J: Centralizing automorphism of prime rings, Canad. Math. Bull.Vol.19,(1976), 113-115.

[15] Nobusawa.N: On the Generalization of the Ring Theory, Osaka J.Math.,Vol. 1,(1964),81-89.

[16] Serif Yenigul.M, Argac Nurcan: Ideals and Symmetric bi-derivations of prime and semi-prime rings, Math. J. Okayama Univ, Vol. 35, (1993), 189-192.

[17] Vukman.J: Symmetric bi-derivations on prime and semi-prime rings, Aequations Math, Vol.38, (1989), 245-254.

[18] Vukman.J: Centralizers in prime and semiprime rings, Comment. Math.Univ. Carolinae, Vol. 38,(1997),231-240.

[19] Zalar.B: On centralizers of semiprime rings, Comment.Math. Univ. Carolinae, Vol. 32,(1991),609-614.